
cisco Live!
ALL IN

The bridge to possible

Public Key Cryptography

From RSA and EC to Post-Quantum

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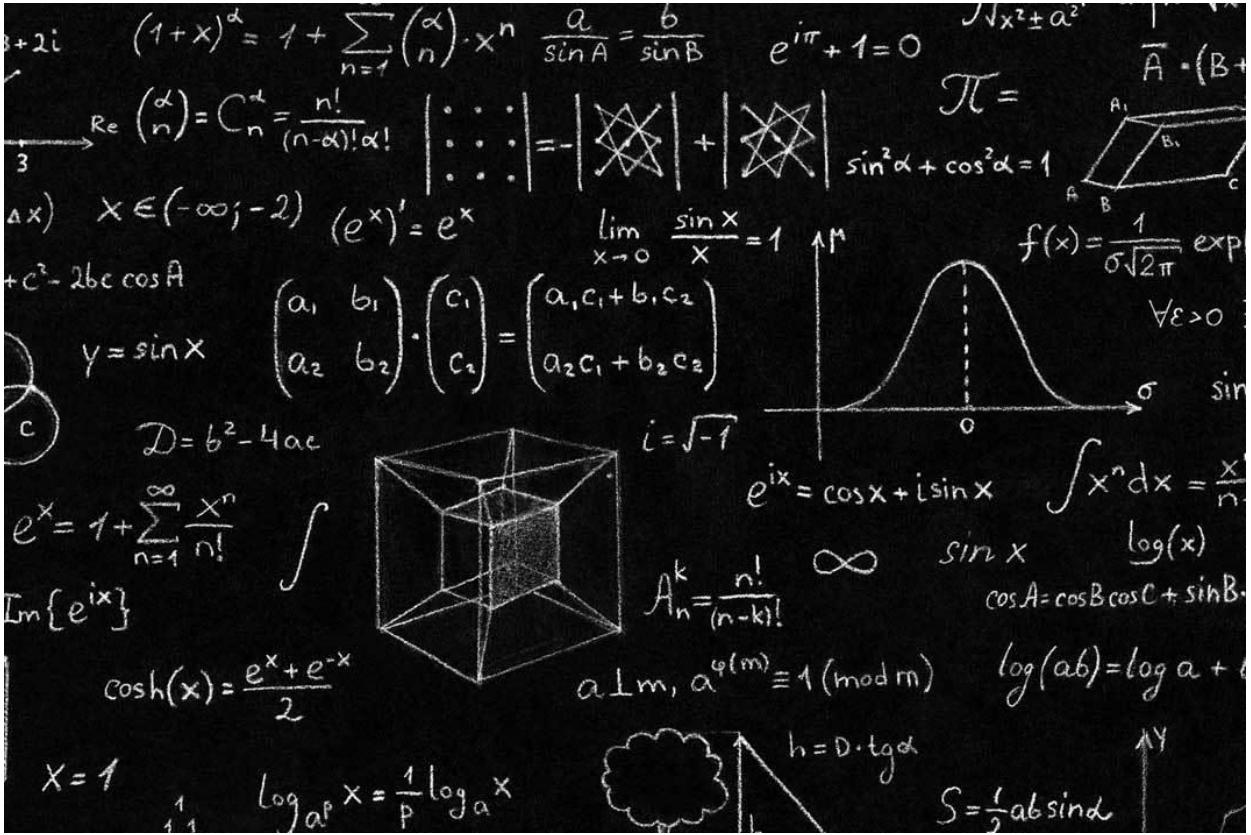
BRKSEC-3129

Agenda

- A Brief Introduction
- MODP: Multiplicative Group of Integers Modulo P
- ECC: Elliptic Curve Cryptography
- Enters The Quantum Computer
- Lattice Based Cryptography
- Conclusion and Recommendations

Introduction

Today is a bout making math fun!



Cryptographic Mechanisms

Encryption

Signatures

Data Authentication (HMAC)

Random Number Generation

Key Establishment

Hashing

Today – Suite B

Authenticated Encryption	AES-GCM
Authentication	HMAC-SHA-2
Key Establishment	ECDH
Digital Signatures	ECDSA
Hashing	SHA-2
Entropy	SP800-90
Protocols	TLSv1.2, IKEv2, IPsec, MACSec

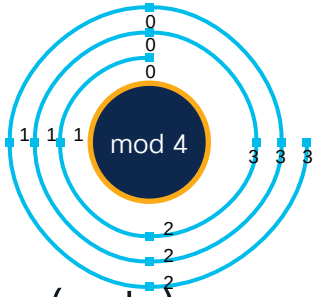
MODP Multiplicative Group of Integers Modulo P

RSA

- Rivest, Shamir, Adleman (1977)
  - Patented but expired => no more royalty
- Public key cryptosystem
- Variable key length (usually 512-2048 bits)
- Based on the (current) difficulty of factoring very large numbers

Modular Arithmetic

- Modulo is like a clock



- $b^x \bmod n = r$ also written as $b^x \equiv r \pmod n$
  - b is the base
  - x is the exponent
  - n is the modulus
  - r is the remainder
- Knowing b, x & n, it is very easy to compute r
- Knowing x, r & n, it is very difficult to compute $b = \sqrt[x]{r} \bmod n$ aka the RSA problem
- Knowing b, r & n, it is very difficult to compute $x = \log_b(r) \bmod n$ aka the discrete log problem

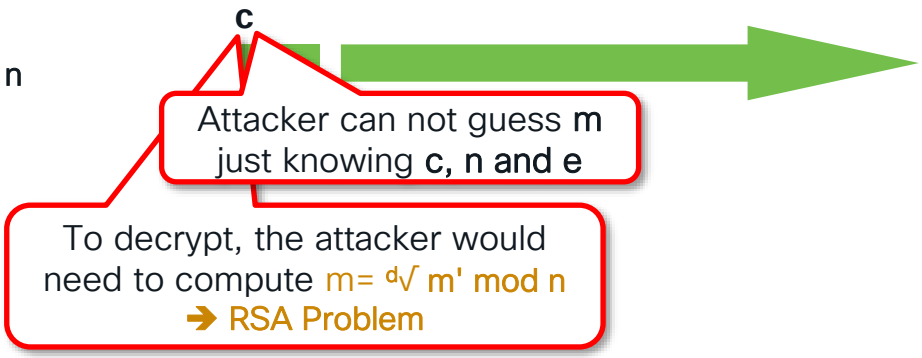
unless there are trapdoors

Encryption with Modular Arithmetic

Alice

Must send a private message **m**

Takes **n** & **e** from Bob
(we assume m < n)
Computes **c = m^e mod n**



Bob

Selects three numbers **n, d & e**
n & **e** are **public**, **d** is **secret**
e, d are chosen such as ed ≡ 1 mod n

Computes **m' = c^d mod n**

$m' = c^d \bmod n$
$= (m^e)^d \bmod n$
$= m^{ed} \bmod n$
$= m^1 \bmod n$
$= m$

➔Bob has reversed the operation !!
➔Bob knows d but nobody else…
➔We have an encryption scheme

Signature with Modular Arithmetic

Alice

Takes **n** & **e** from Bob

Computes **m' = c^e mod n**

m' = c^e mod n
= $(m^d)^e$ mod n
= m^{de} mod n
= m^1 mod n
= m mod n
= m

➔Bob must have sent the **c,m**

Bob

Selects three numbers **n, d & e**
n & **e** are **public**, **d** is **secret**
e, d are chosen such as ed ≡ 1 mod n

Must send a signed message **m**

Computes **c = m^d mod n**
(we assume m < n)

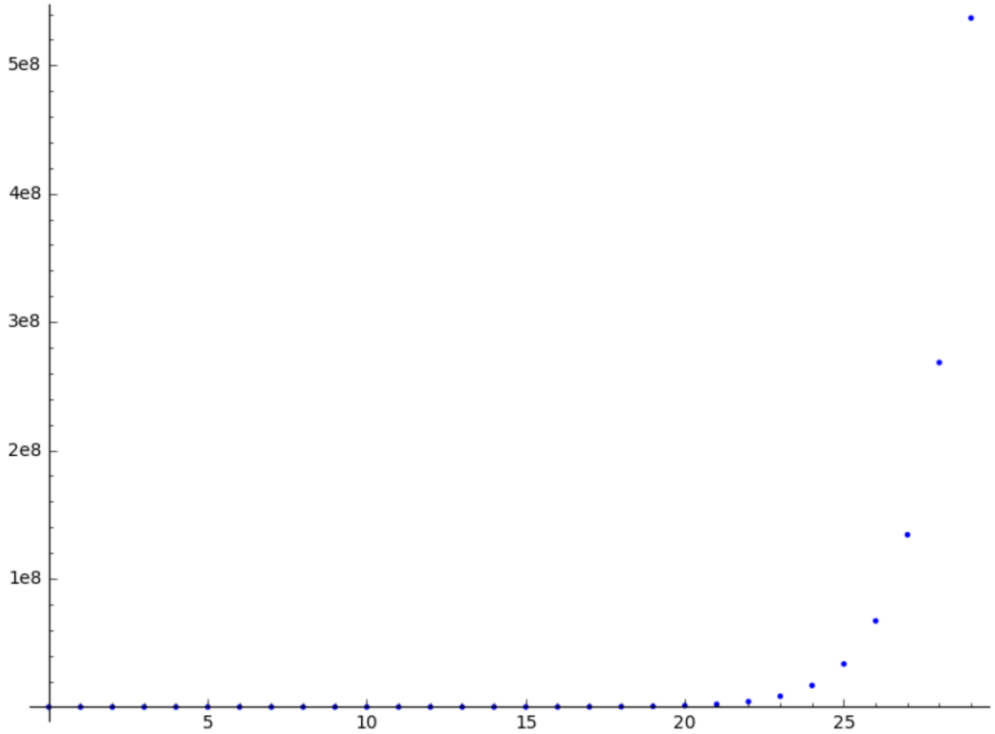
c, m

Attacker can not guess **d** just knowing **m, n and e**

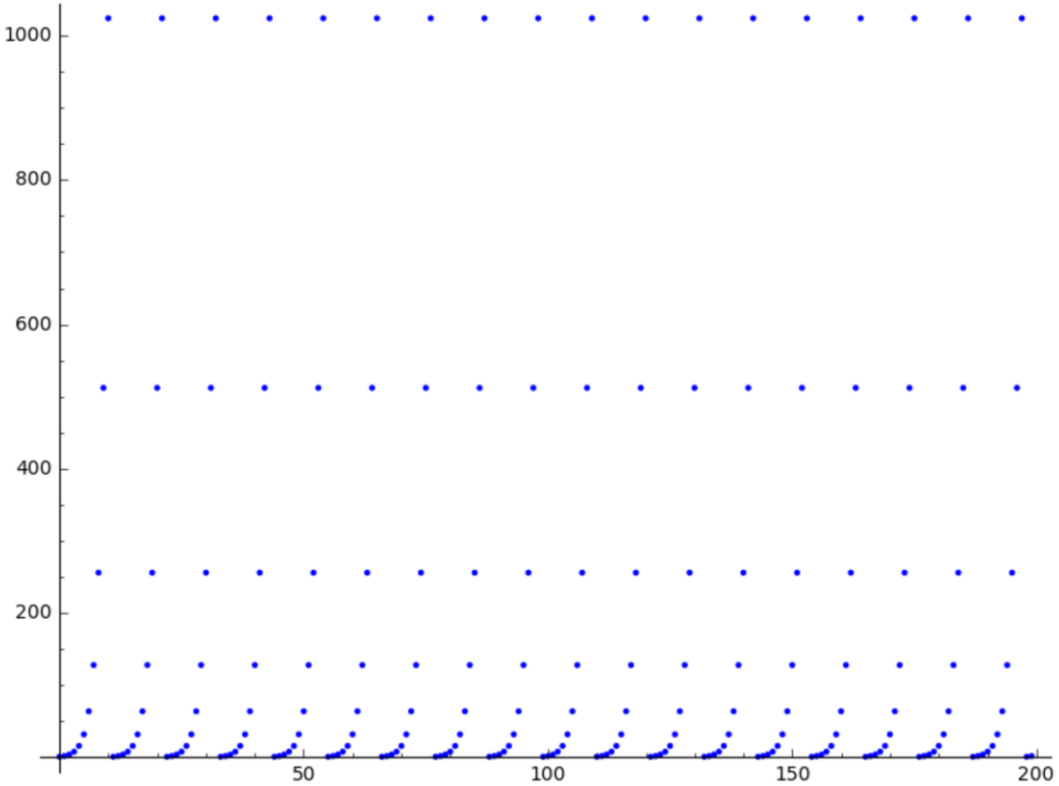
To forge the signature, the attacker would need to compute
$d = \log_e(m') \bmod n$
➔ Discrete Logarithm Problem

Now how can we find such e, d and n ?

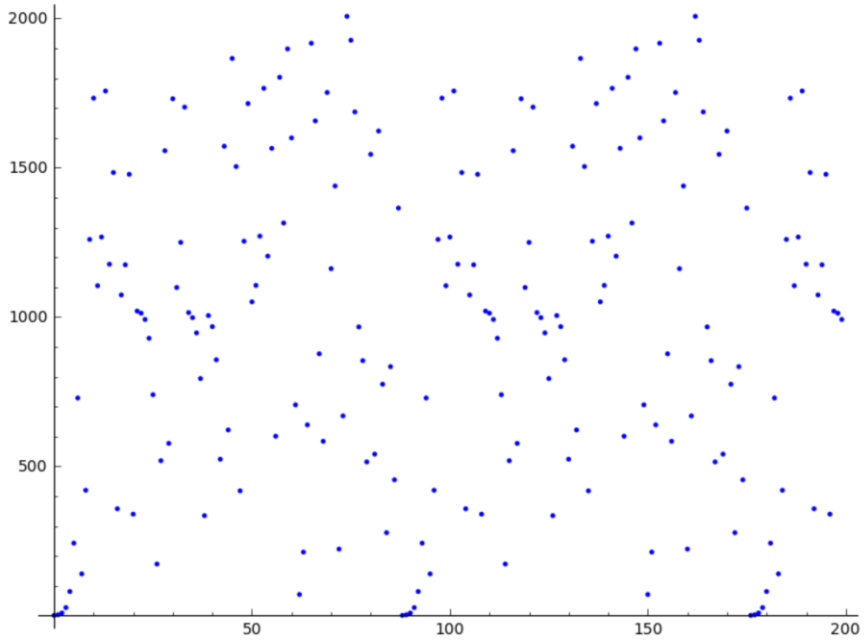
Regular Exponentiation –Dichotomy to reverse

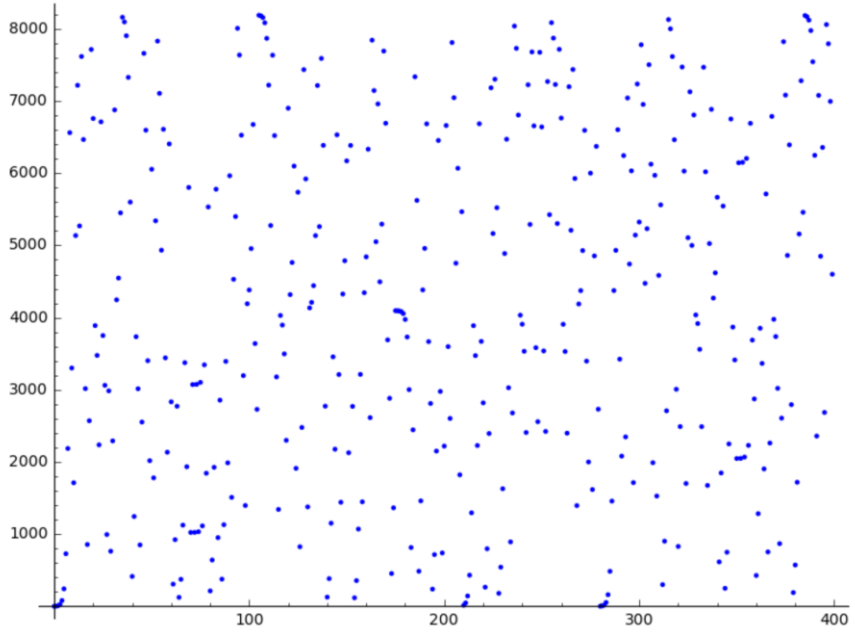


MODP Exponentiation – dichotomy is broken

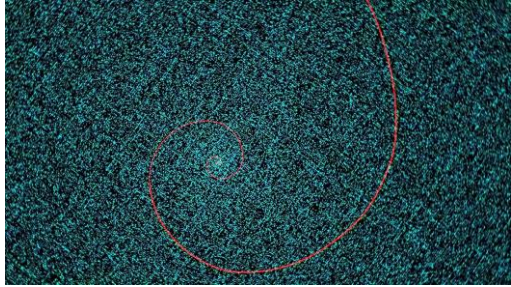


With well chosen numbers...





About Prime Numbers



picture: Khan Academy

- A number is prime if it can be divided by one or itself
- A number is composite is it can be divided by 2 or more prime numbers
  - Factorization is a hard problem. Best algorithm yields $O\left(\exp\left(\left(\frac{64}{9}b\right)^{\frac{1}{3}}(\log b)^{\frac{2}{3}}\right)\right)$.
- Fundamental Theorem of Arithmetic: a given number has a single factorization
- Euclid's theorems: there are infinitely many primes
  - prime density (ratio of primes per composite up to x) is 1/ln(x)
  - density drops off rapidly in the beginning but very slowly after a few powers of 10
  - π(x) = x/ln(x) : number of primes < x
- Euler's φ(n) function or Euleur's totient
  - # of integers in [1,n] that are relatively prime to n: |k ∈ [2,n] | GCD(k,n) = 1|, {1}
  - 2 numbers are coprime if they share no factor other than 1.
  - Property: *n1, n2*, GCD(*n1,n2*)=1 ➔ φ(*n1*n2*)= φ(*n1*)* φ(*n2*) – Totient is multiplicative
  - if x is prime ➔ φ(*n*) = n-1 since 1…n-1 coprime with n and n divisible by itself
- Euler's theorem: $m^{\varphi(n)} \equiv 1 \pmod{n}$ if *m* and *n* are co-prime.

RSA keys – finding e,d,n | $m^{ed} \equiv m \pmod n$

- Choose two distinct prime numbers *p, q* and hide them forever!
- $n = p.q$ ➔ n is hard to factor if p & q are very large
- $\varphi(n) = n-(p+q-1)$
  - p & q are prime ➔ $\varphi(p)=p-1$ $\varphi(q)=q-1$
  - $\varphi(n) = \varphi(pq) = \varphi(p)\,\varphi(q) = (p-1)(q-1) = n-(p+q-1)$
- Final steps — Euler theorem...
  - $1^k = 1$ ➔ $(m^{\varphi(n)})^k \equiv 1^k \pmod n$ ➔ $m^{k\varphi(n)} \equiv 1 \pmod n$
  - $1m = m$ ➔ $m\, m^{k\varphi(n)} \equiv m \pmod n$ ➔ $m^{k\varphi(n)+1} \equiv m \pmod n$
  - we look for e,d,n such that $m^{ed} \equiv m^{k\varphi(n)+1} \equiv m \pmod n$ ➔ $ed = k\,\varphi(n) + 1$
  - ➔ $d = \frac{k\,\varphi(n)+1}{e} = \frac{k\,(n-(p+q-1))+1}{e}$
- Select **e**, small integer and **k** such that GCD(d, $\varphi(n)$) = 1 (i.e. d & $\varphi(n)$ are co-prime)
  - **e** is usually 3 or 65537
  - adjust **k** to make **d** an integer

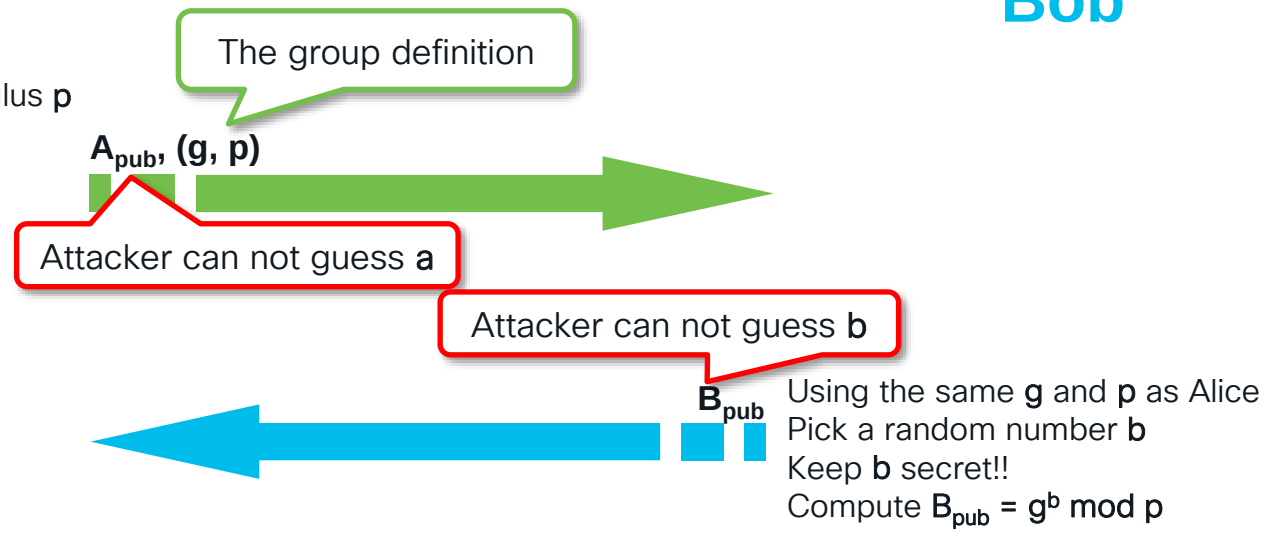
m – arbitrary message
n – the modulus
e – the public key
d – the private key

DH –Diffie–Hellman

Alice

Select a generator **g** and a modulus **p**
Pick a random number **a**
Keep **a** secret!!
Compute $A_{pub} = g^a \bmod p$

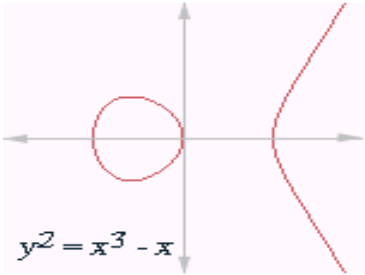
Bob

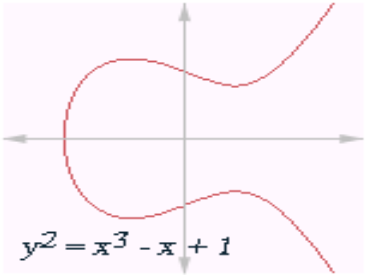


$Secret_{Alice} = (B_{pub})^a \bmod p$ = $Secret = g^{a.b} \bmod p$ = $Secret_{Bob} = (A_{pub})^b \bmod p$

ECC
Elliptic Curve
Cryptography

What is an elliptic curve ?

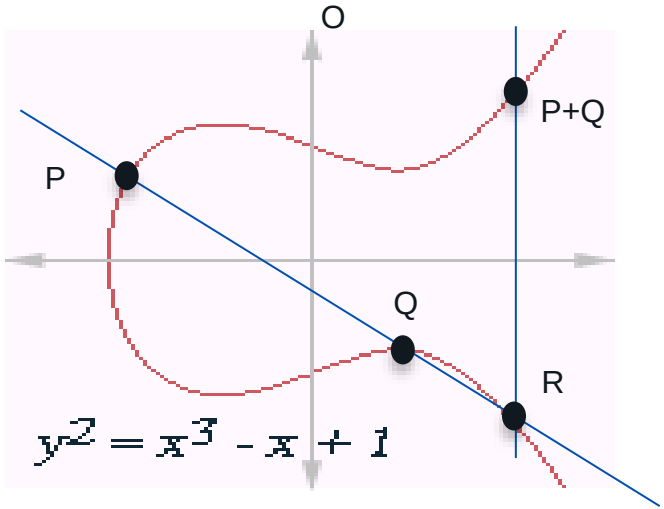


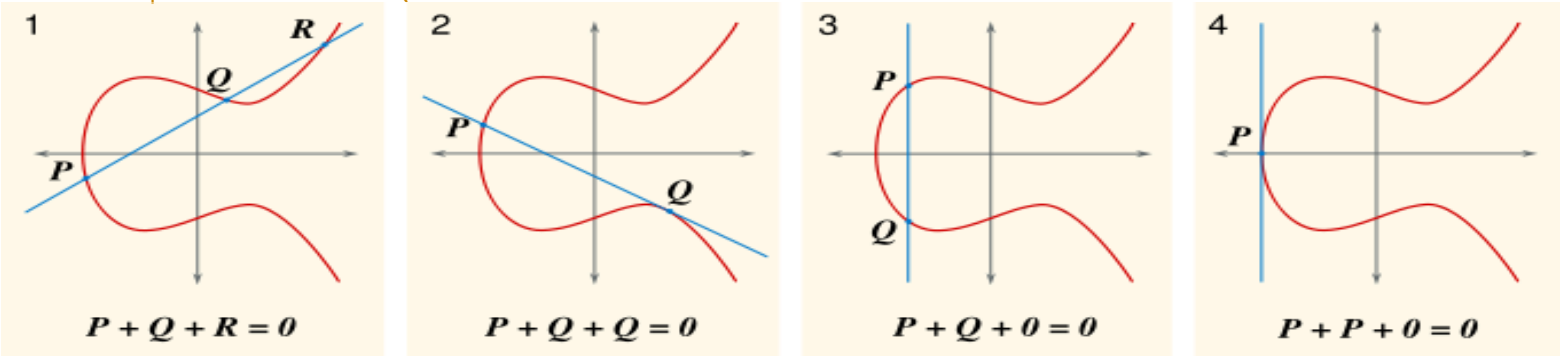


- A curve of general equation $y^2=x^3+ax+b$
  - It MUST be a smooth curve
  - Its discriminant MUST BE NON ZERO: $D = 4A^3 + 27B^2$
- The Elliptic Curve is the set of points
  - that satisfy the equation of the curve (ie. that “belong” to the curve)
  - Plus a special point at infinity that we call O (the letter O)

Elliptic Curve Addition

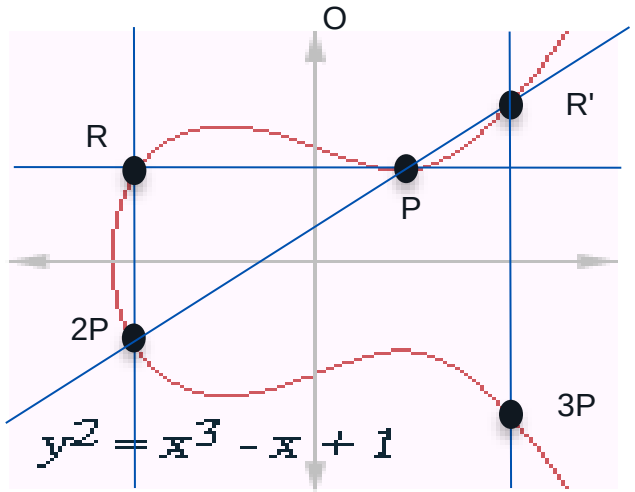
- Let P and Q be two points on the curve
- A line (P,Q) cuts the curve at a third point R
  - If the line is parallel to the Y axis, this point is O
  - If the line is tangent to the curve, the tangent point is counted twice
- The group operator + is defined such as
  - P+Q+R = O; O is the identity
- The reflected point from R is P+Q



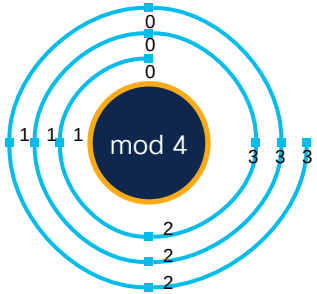


The scalar multiplication n*P

- Let’s start with P+P = 2*P
- For drawing (P,P)
  - draw a tangent to the curve → R
  - (O,R) cuts in P+P=2P
- This is a scalar multiplication
  - One can derive 3P = 2P+P, 4P = 3P+P,...nP = (n-1)P+P



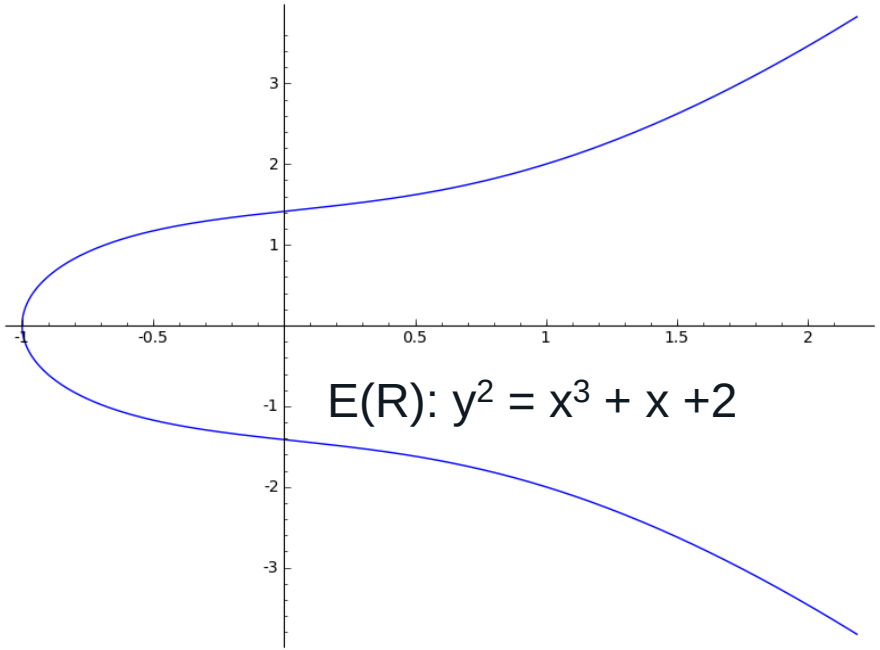
Fast Forward – the finite fields F_m & F_{2^k}



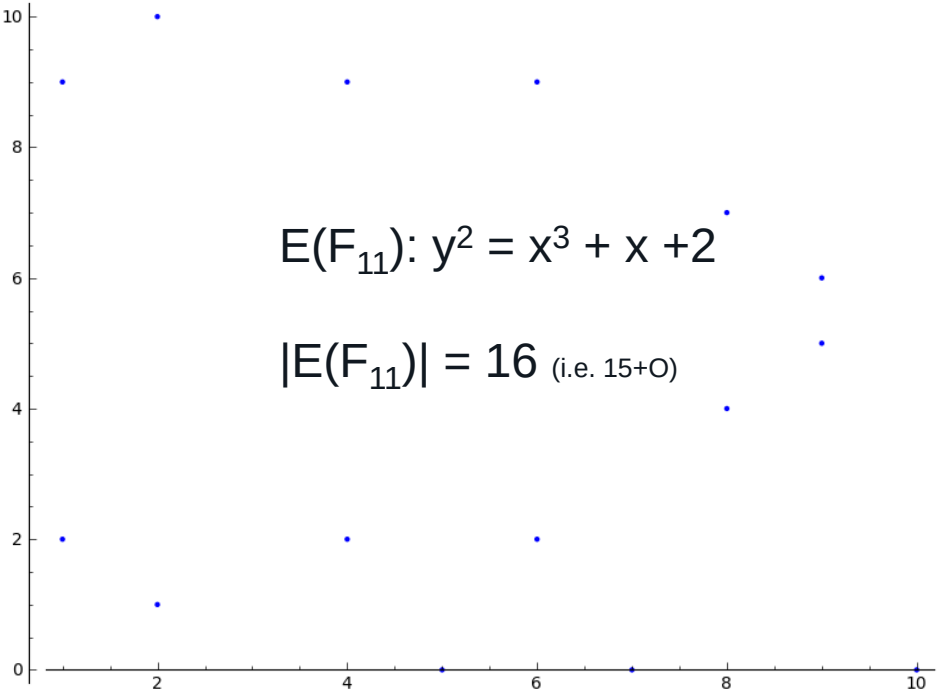
- Remember… modulo arithmetic
- Galois Field = Finite Field
- Let E be an elliptic curve defined over a finite field F_m (modulo m):
  - **$E(Fm):\{\infty\} \cup \{(x,y) \text{ in } F_m x F_m \mid y^2=x^3+ax+b \text{ , } a,b \text{ in } F_m\}$**
  - $E(F_m)$ is the set of points whose coordinates belong to $F_m x F_m$ and satisfy the equation + point at infinity
  - The set along group operations (+, x) seen before form an Abelian Group under multiplication ➔ a field.
  - For cryptography, m should be a prime number

- It **seems (seemed ?)** more computationally efficient if **$m = 2^k-1$** yielding the notation **F_{2^k}**
  - Multiplication supposed to be more efficient ➔ very important for ECDH and ECDS
  - In this case, the Koblitz curve is used: **$y^2 + xy = x^3 + ax^2 + 1$** where **$a=0$** or **$a=1$**
  - For cryptography, k should be a prime number
  - m should remain a prime – it would be called a Mersenne Prime
  - **There is debate about the actual security and efficiency of these curves**!
- The **order of a group *G*** is the **cardinality** of that group written ***ord(G)*** or ***|G|***.
- The order of a **point *P* in a group *G*** is the value *n* such that *n*P = O* written ***ord(p)*** or ***|p|***

Example Curve



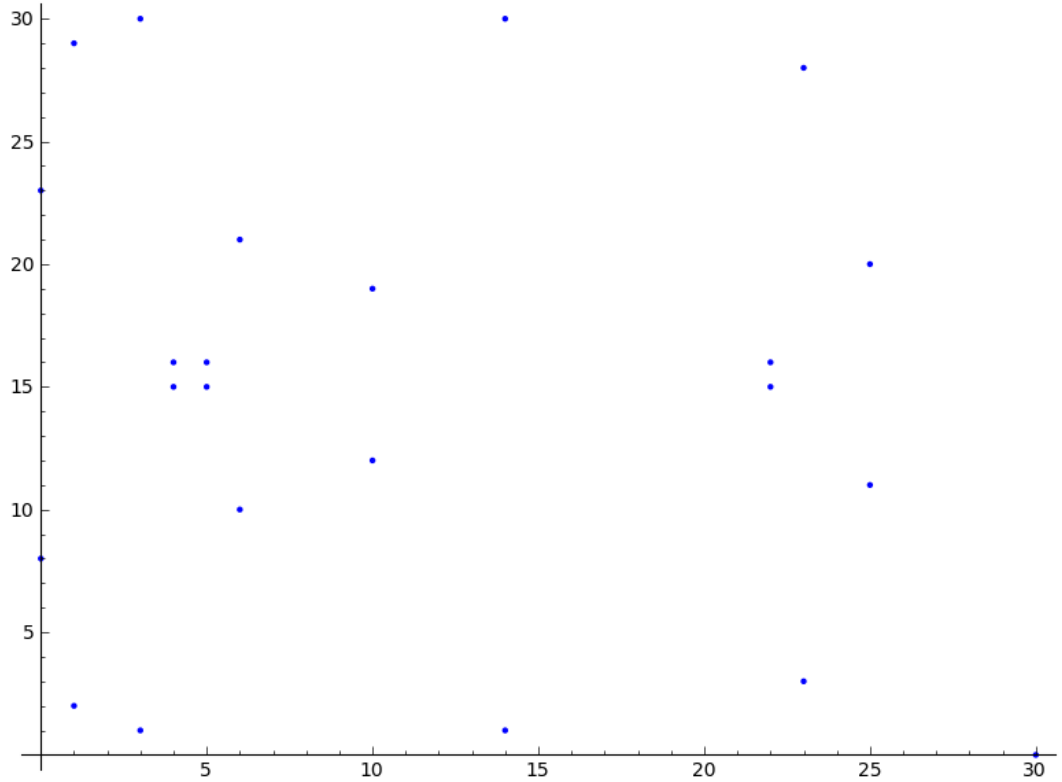
$E(R)$: $y^2 = x^3 + x + 2$



$E(F_{11})$: $y^2 = x^3 + x + 2$

$|E(F_{11})| = 16$ (i.e. 15+O)

Example on F_{31} – Complexity Increases

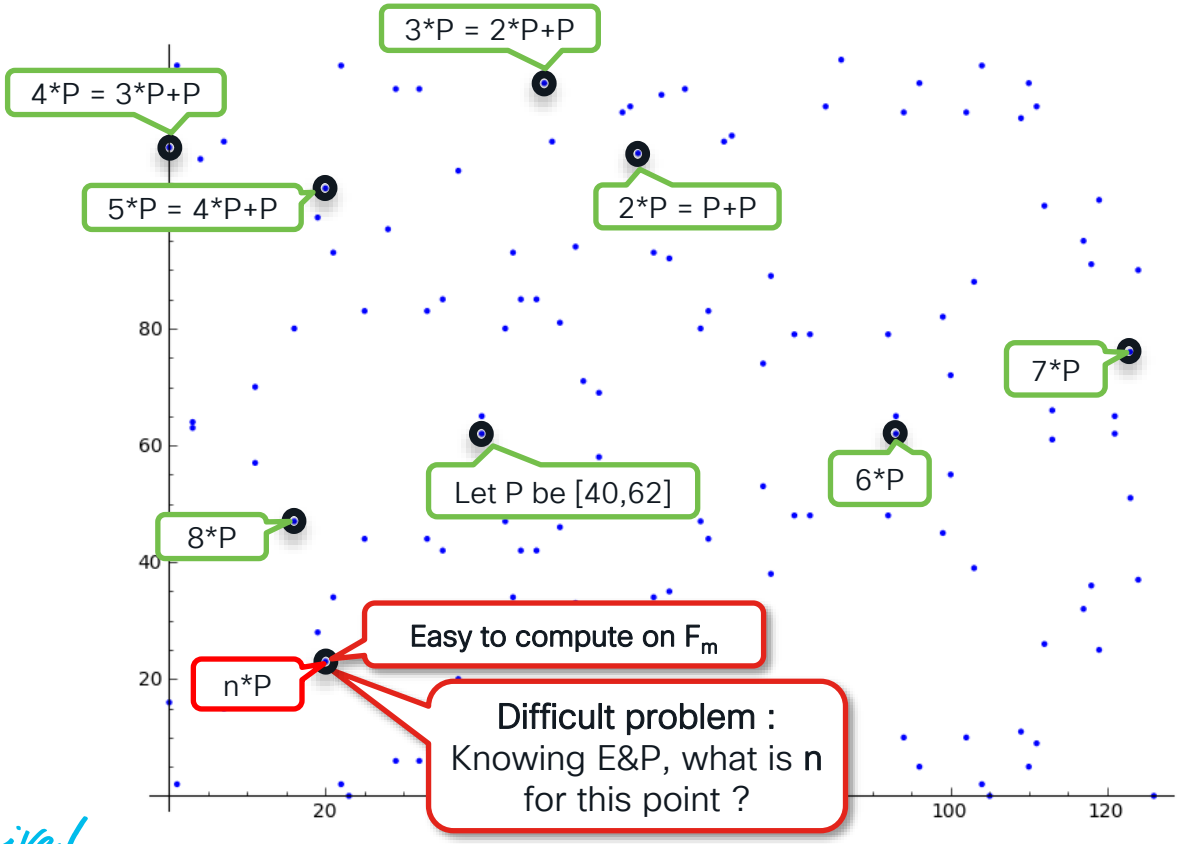


$m = 2^5-1 = 31$

$E(F_{31})$: $y^2 = x^3 + x + 2$

$|E(F_{31})| = 24$

The same on F_{127} – Complexity Further Increases

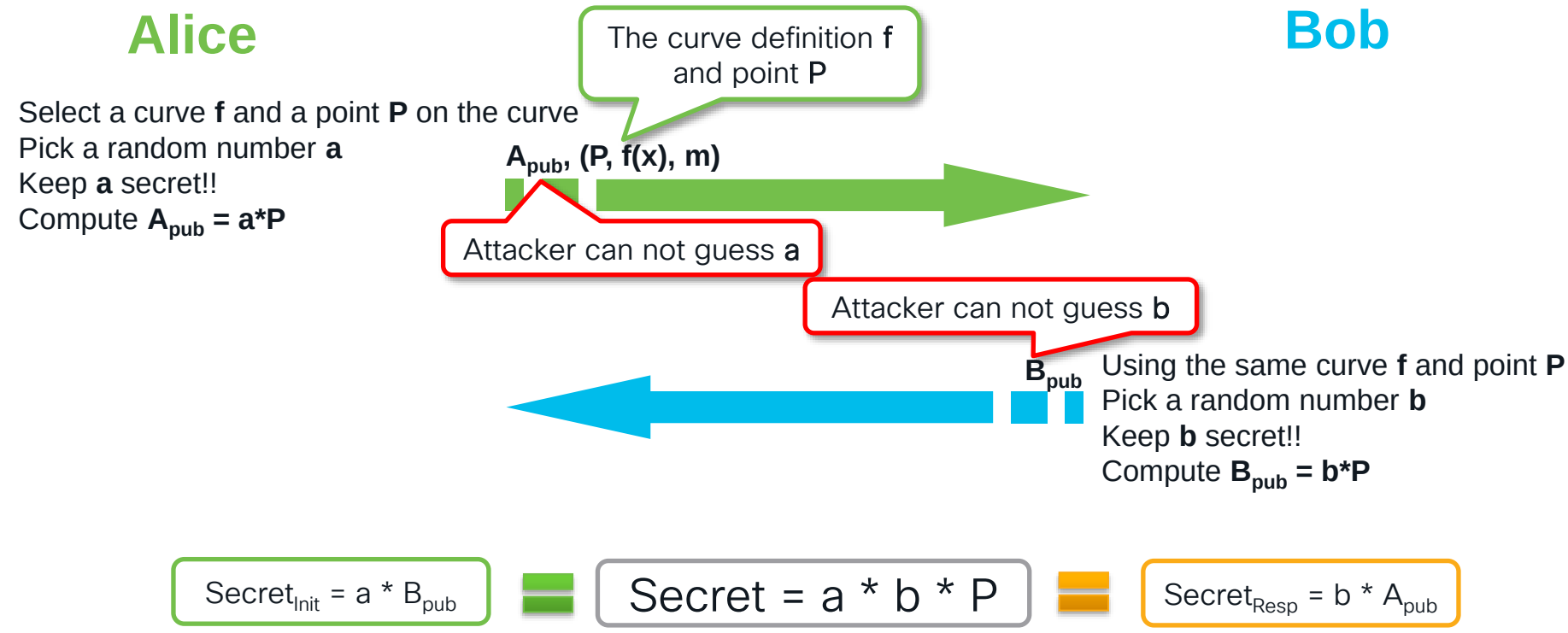


$m = 2^7-1 = 127$

$E(F_{127})$: $y^2 = x^3 + x + 2$

$|E(F_{127})| = 136$

ECDH – Elliptic Curve Diffie-Hellman



Representation of Elliptic Curves

- Elliptic curve domain parameters
  - (p, a, b, G, n, h) for a curve over a prime field F_p
  - (m, f(x), a, b, G, n, h) for a curve over a binary field F_{2^m}
- Where
  - **p** is the prime modulus
  - **G** is the generator (base point) of the curve
  - **n** is the order of G. i.e **n*G=O**
  - **a, b** are the coefficient of $y^2 + xy = x^3 + ax + b \pmod p$
- Who defines elliptic curves ?
  - National Institute of Standards and Technology (NIST)
  - American National Standard Institute (ANSI)
  - Agence Nationale pour la Securité des Systèmes Informatiques(ANSSI)
  - Institute of Electrical and Electronics Engineers (IEEE)
  - Certicom
  - Brainpool ECC

P-256 from "NIST routines"

4.3 Curve P–256

4.3.1 Parameters

The curve P–256 is given by the following parameters (see also [FIPS186-2]). The prime $p_{256} = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$:

$p_{256} =$ 115792089210356248762697446949407573530086143415290314195533631308867097853951

in hexadecimal form:

$p_{256} =$ ffffffff 00000001 00000000 00000000 00000000 ffffffff ffffffff ffffffff

The parameter $a = p_{256} - 3$:

$a =$ 115792089210356248762697446949407573530086143415290314195533631308867097853948

in hexadecimal form:

$a =$ ffffffff 00000001 00000000 00000000 00000000 ffffffff ffffffff fffffffc

the parameter b:

$b =$ 41058363725152142129326129780047268409114441015993725554835256314039467401291

in hexadecimal form:

$b =$ 5ac635d8 aa3a93e7 b3ebbd55 769886bc 651d06b0 cc53b0f6 3bce3c3e 27d2604b

Base point G:

$x_G =$ 48439561293906451759052585252797914202762949526041747995844080717082404635286

$y_G =$ 36134250956749795798585127919587881956611106672985015071877198253568414405109

in hexadecimal form:

$x_G =$ 6b17d1f2 e12c4247 f8bce6e5 63a440f2 77037d81 2deb33a0 f4a13945 d898c296

$y_G =$ 4fe342e2 fe1a7f9b 8ee7eb4a 7c0f9e16 2bce3357 6b315ece cbb64068 37bf51f5

in hexadecimal form with X9.63 compression (lead byte 02 if y_G is even, 03 if y_G is odd):

$\overline{G} =$ 00000003 6b17d1f2 e12c4247 f8bce6e5 63a440f2 77037d81 2deb33a0 f4a13945 d898c296

Order q of the point G (and of the elliptic curve group E):

$q =$ 115792089210356248762697446949407573529996955224135760342422259061068512044369

hexadecimal form:

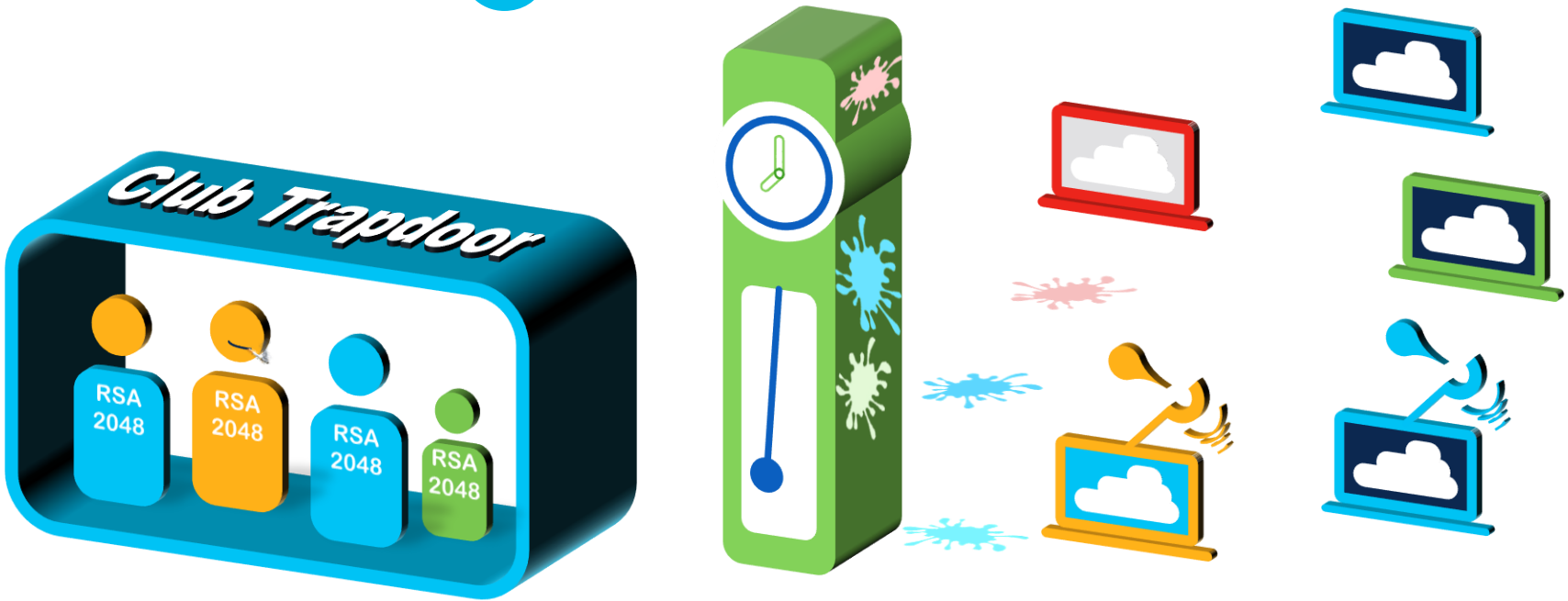
$q =$ ffffffff 00000000 ffffffff ffffffff bce6faad a7179e84 f3b9cac2 fc632551

Enters the Quantum Computer

Today's Cryptography Temporal Defense

TIME PROTECTS PUBLIC KEYS (Until Y2Q)

Public Key = Prime 1 x Prime 2



Shor's Factoring Algorithm

Problem: For a given "N" find a "p" between "1" and "N" that divides "N"

$N = p_1 * p_2$

Peter Shor

1994

Part A
Classic
Reduction of factoring problem to period finding problem

Part B
Quantum
Solve period finding problem using Quantum Fourier Transform

(Probabilistic Algorithm)

(Adjust it to taste if it's wrong)

Shor's algo converts exponential complexity to polynomial complexity

$x^N \rightarrow N^x$ where N is the number of bits

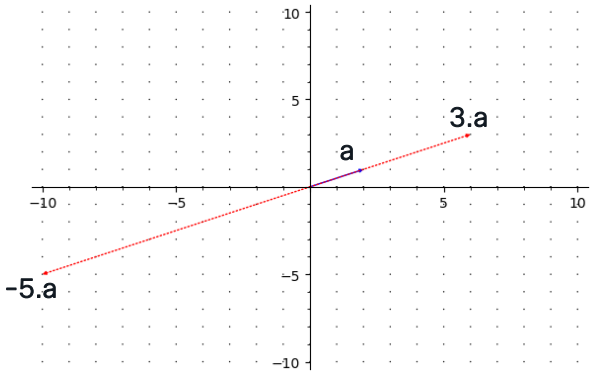
Lattice Based Cryptography

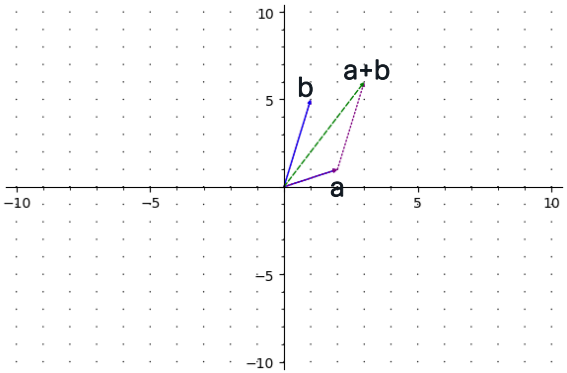
LBC, LWE, NTRU

Vectors

Commonly denoted $\boldsymbol{v}$ or $\vec{v}$
We will use ***v*** *or* $\boldsymbol{v}$

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$



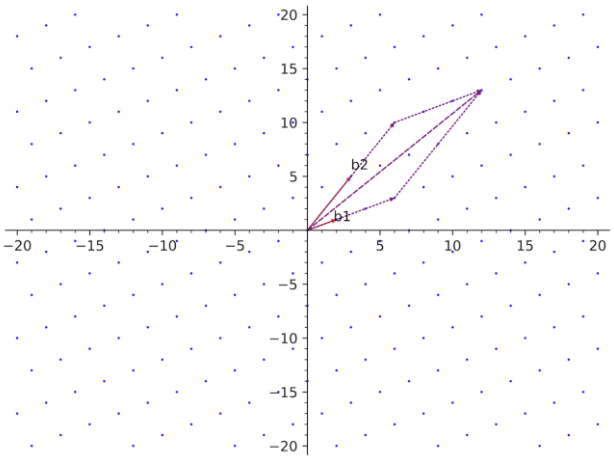


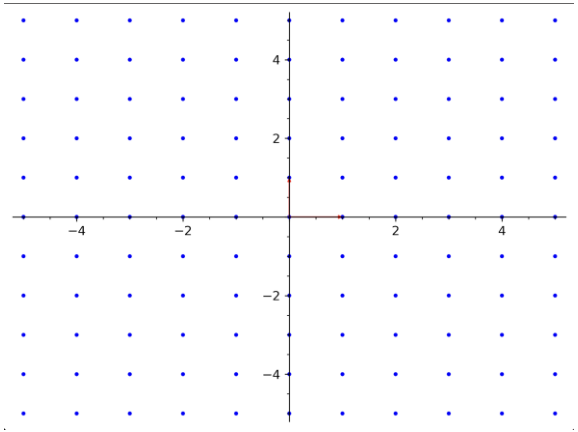
Operation	Formula	Result type
addition	$\boldsymbol{a}+\boldsymbol{b} = (a_1+b_1,\dots,a_2+b_2)$	Vector
Scalar multiplication	$x.\boldsymbol{a} = x.a_1+\dots+x.a_n$	Vector
Inner product	$\boldsymbol{a}.\boldsymbol{b} = a_1b_1+\dots+a_nb_n$	Scalar (number)

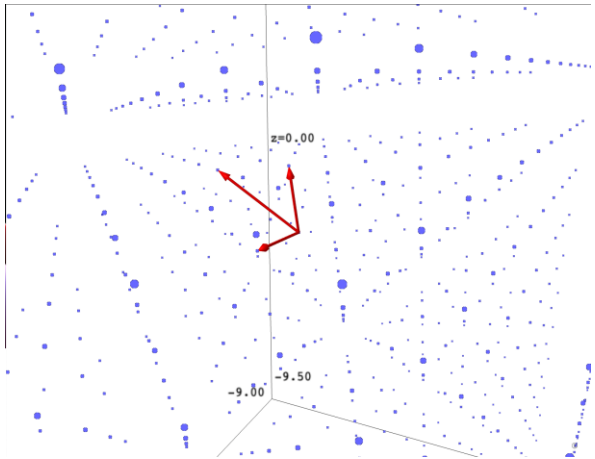
This is not a lattice

What is a Lattice ?

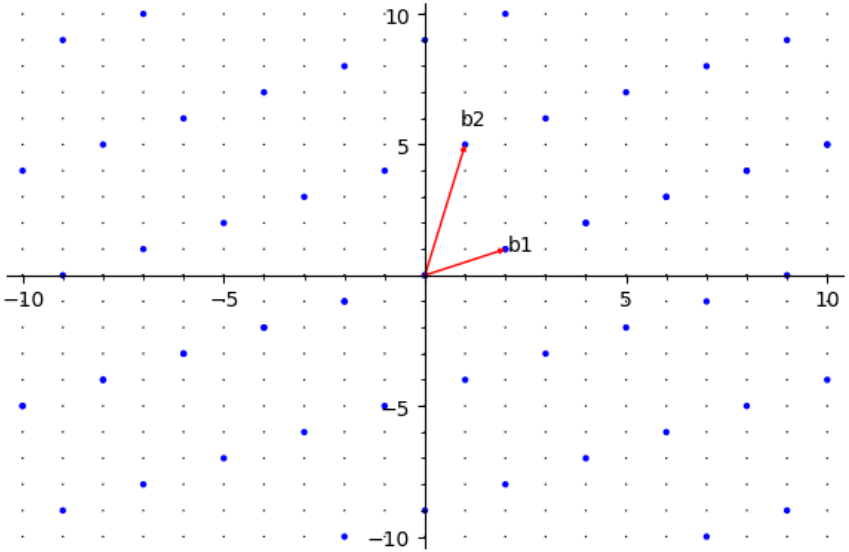
- A periodic “grid” in $\mathbb{Z}^m$
- All integer linear combinatons of n basis vectors $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}$
- Basis $\mathbf{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_m}\}$
- Lattice $\mathcal{L} = \sum_{i=1}^{m} a_i \cdot \boldsymbol{b}_i \ , a_i \in \mathbb{Z}$

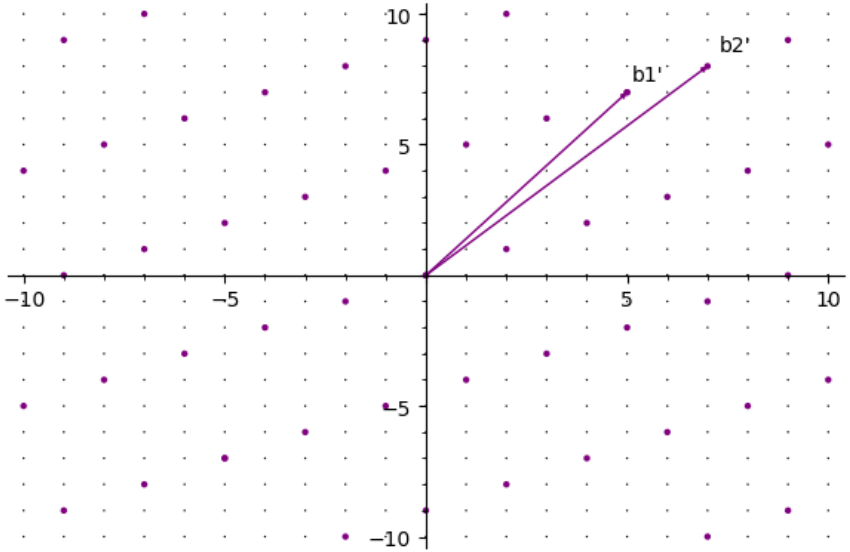






Good Basis, Bad Basis

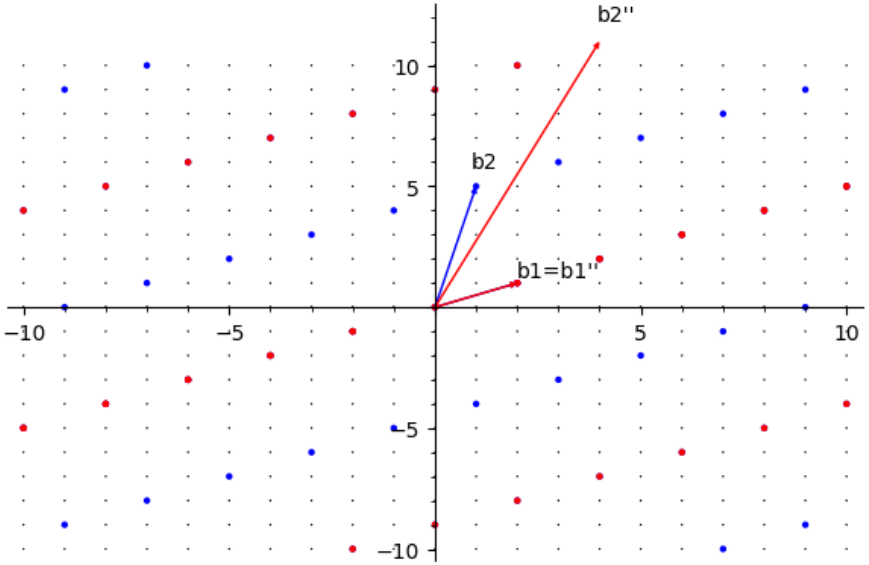




B= {$\mathbf{b}_1$, $\mathbf{b}_2$}: good basis
Short, almost perpendicular vectors

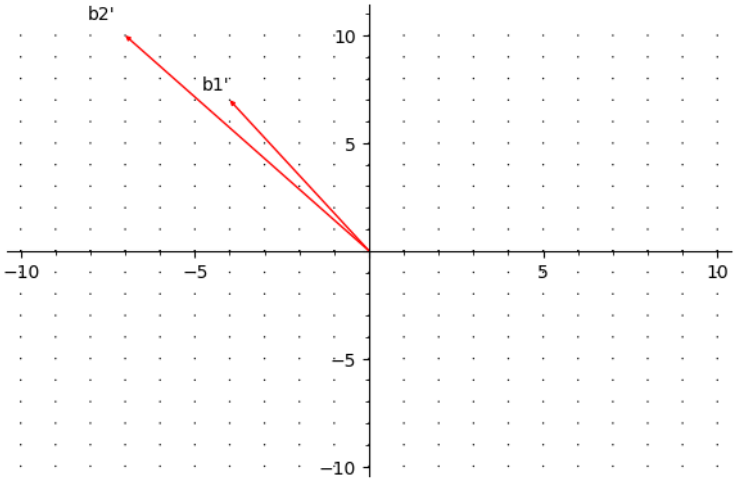
B= {$\mathbf{b}_1$', $\mathbf{b}_2$'}: bad basis
Long, not very perpendicular vectors

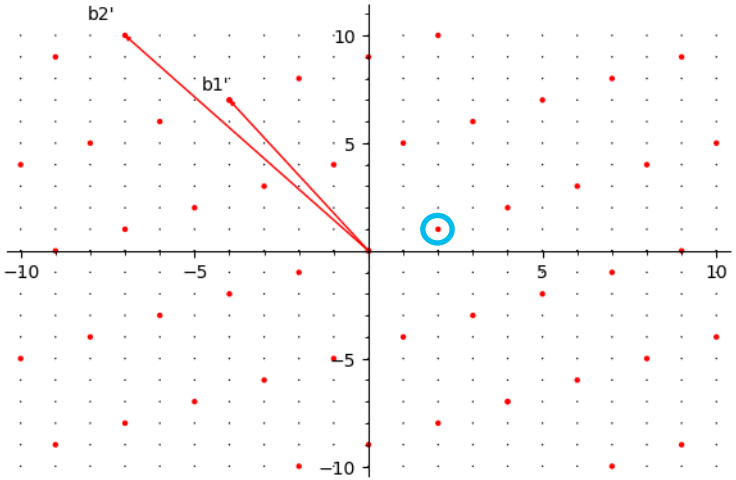
Not a basis



B= {b_1'', b_2''}: not a basis
The lattices do not overlap fully

Short Vectors is a Hard Problem



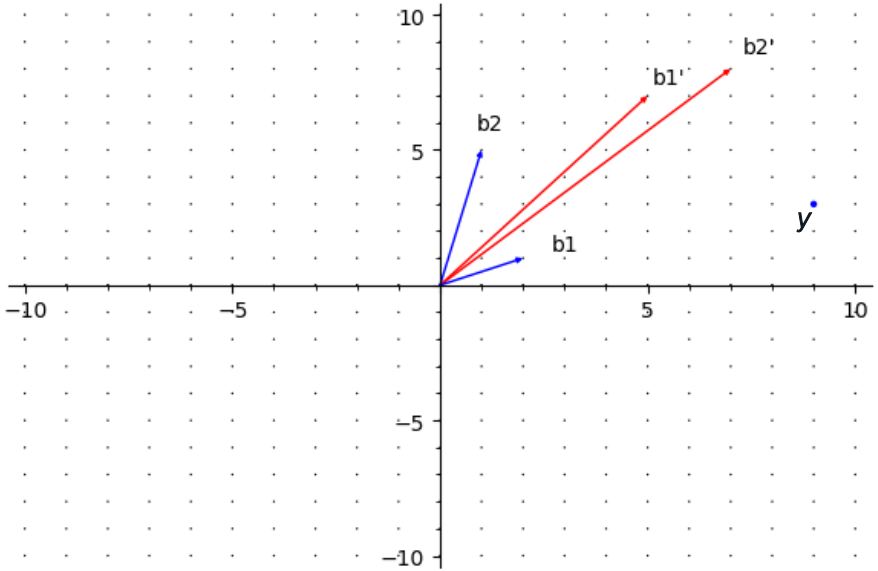


Given this base, what is the shortest possible non-trivial vector ?

Surprise!

Closest Vector Problem is a Hard Problem

What is the closest lattice vector to y ?



Babai's round-off algorithm:

$$\boldsymbol{v} = B.\lfloor B^{-1}.\boldsymbol{y} \rceil$$

Theorem:

$$\|\boldsymbol{v} - \boldsymbol{y}\| \leq \frac{1}{2}\sum \|\boldsymbol{b_i}\|$$

In clear:
Better base ➔ Closer vector

Short Integer Solution & Learning With Errors

SIS

$A\boldsymbol{z} = 0$ with 'short' $\boldsymbol{z} \neq 0$

Average case SVP
(Bounded Distance Decision)

$$\mathcal{L}^{\perp}(A) = \{z \in \mathbb{Z}^m : A\boldsymbol{z} = 0\}$$

LWE

$(A, \boldsymbol{b}^t = \boldsymbol{s}^t A + \boldsymbol{e}^t)$ vs. $(A, \boldsymbol{b}^t)$

Average case BDD
(Bounded Distance Decision)

$$\mathcal{L}\,(A) = \{\boldsymbol{z}^t \equiv \boldsymbol{s}^t A \bmod q\}$$

These are other ways to define a lattice

Example Only!

Goldreich, Goldwasser, Halevi Cryptosystem

Alice

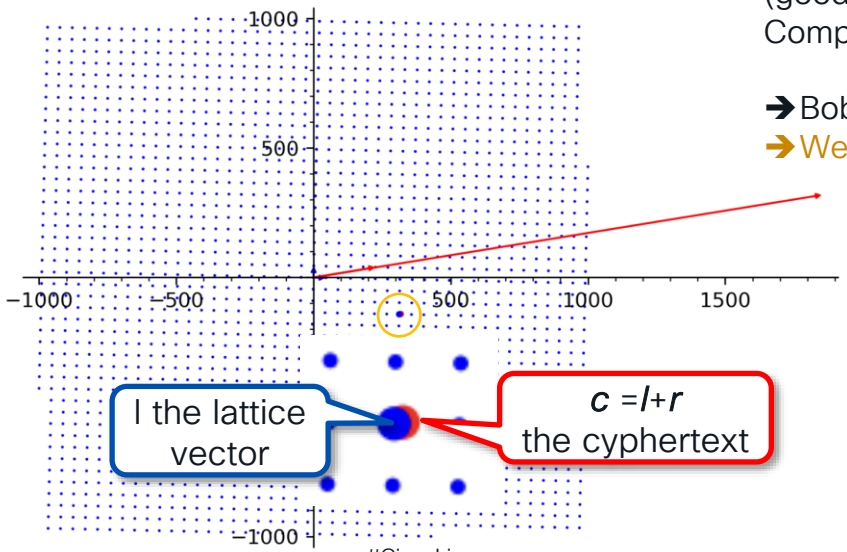
Must send a private message **m**

Takes **V'** from Bob
Computes $l = m * V'$
Generates an error vector r (n)
Computes $c = l + r$

Quantum secure
... but broken ☹

Eve can not guess *m* because of the Closest Vector Problem (bad basis)

c



Bob

Generates **V** (nxn), a good basis
Publishes **V'** (nxn), a bad basis

Applies Babai on (V, *c*) → finds *l*
(good basis → accuracy)
Computes $m = l * V'^{-1}$

➔Bob has reversed the operation !!
➔We have an encryption scheme

Post Quantum Algorithm Selection – Round 3

Type	Name	Math
Pub Key Encr and Key Exchange	Classic McEliece	Linear Code
Pub Key Encr and Key Exchange	CRYSTAL-KYBER	Lattice LWE (CVP)
Pub Key Encr and Key Exchange	NTRU	Lattice SVP
Pub Key Encr and Key Exchange	SABER	Module Learning with Rounding
Digital Signature Algo.	CRYSTAL-DILITHIUM	Lattice LWE (CVP)
Digital Signature Algo.	FALCON	Lattice NTRU(SVP) + FFT
Digital Signature Algo.	Rainbow	Mutlivariate Public Key

Recommendations and Conclusion

Reassurance

- Crypto is not broken; it evolves and so do attackers.
- These are good news! The more research, the more insight.
- Lattices represent an interesting development but are not alone.

Evolve your systems as new recommended algorithms are released !

A Short Bibliography

- NIST SP 800-90A : Recommendations for Random Number Generation Using Deterministic Random Bit Generators
- NIST SP 800-38D : Recommendation for Block Cipher Modes of Operation: Galois/Counter Mode (GCM) and GMAC
- NIST SP 800-56A (R2): Pair-Wise Key Establishment Schemes Using Discrete Logarithm Cryptography (i.e. DH, ECDH + key derivation methods)
- NIST 800-131Ar1: Transitions: Recommendations fro Transitioning the Use of Cryptographic Algorithms and Key Lengths
- NIST FIPS 140-2: Security Requirements for Cryptographic Modules
- NIST FIPS 186-4: Digital Signature Standard (DSS) (DSA, RSA (PKCS#1), ECDSA,...)
- NIST FIPS 180-4: Secure Hash Standard (SHA-1, SHA-256,..., SHA-512)
- NIST Routines: https://www.nsa.gov/ia/_files/nist-routines.pdf (Curve P-192, P-224, P-256 etc.)
- Safe Curves: http://safecurves.cr.yp.to
- Transcript Collision Attacks: Breaking authentication in TLS, IKE and SSH: http://www.mitls.org/downloads/transcript-collisions.pdf

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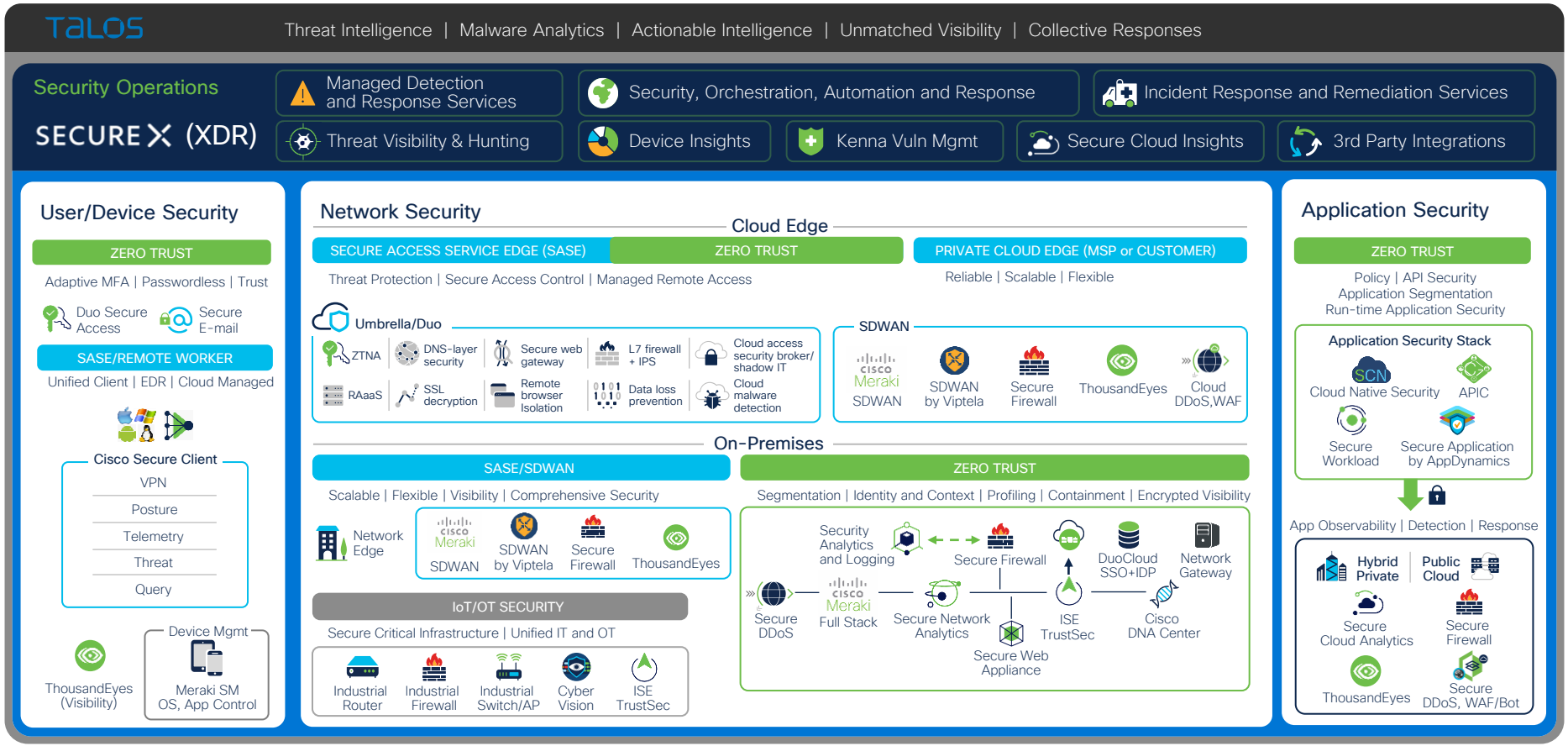

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